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Review on the Ph.D. thesis of Ms Katarzyna Ryszewska

Dear Professor Kotus,

I hereby send you my review on the Ph.D. thesis

A semigroup approach to the space-fractional diffusion and the analysis of fractional Stefan problems

by Ms Katarzyna Ryszewska. I would like to apologise for the slight delay.

Sincerely yours,

Professor Dr. Rico Zacher

Prof. Dr. Rico Zacher
Ulm University, Germany

Review on the Ph.D. Thesis

A semigroup approach to the space-fractional diffusion and the analysis of fractional Stefan problems

by Katarzyna Ryszewska

In recent years, differential equations with fractional derivatives have become a very active and popular field of research, both in physics and mathematics. For example, fractional variants of the classical heat resp. diffusion equation are frequently used to model anomalous diffusion processes. Fractional derivatives lead to new mathematical challenges such as the non-local nature of the corresponding equations, the more involved calculus and the need to work with function spaces of fractional order. The topic of Ms Ryszewska's Ph.D. thesis falls into this area. The dissertation deals with the mathematical analysis of a class of one-dimensional space-fractional diffusion problems and a related fractional, one-phase Stefan problem. The focus lies on existence, uniqueness and regularity of solutions. Ms Ryszewska also considers a time-fractional version of the one-phase Stefan problem and constructs self-similar solutions for both models.

The Ph.D. thesis consists of five chapters. In the following I will go into the individual chapters in more detail.

Chapter 1 provides some motivation from physics and describes the structure of the dissertation in the form of a summary.

Chapter 2 introduces basic concepts from functional analysis (semigroup theory, fractional powers of operators and evolution operators) and fractional calculus (fractional integrals and derivatives, calculation rules and mapping properties) and provides derivations of the space- and time-fractional Stefan models to be studied. The main purpose of Chapter 2 is to provide important tools for the subsequent chapters. Most of the material was taken from the literature. Fractional Sobolev spaces with positive differentiability exponent are introduced in Definition 2.2 via complex interpolation. I must note here that in Remark 2.2 fractional Sobolev spaces of negative order appear and so their definition is not clear at this point. Following Lunardi's book, the notion of an analytic semigroup generated by a sectorial operator is defined in the constructive way, see Definition 2.14. Concerning fractional derivatives, the main objects are the fractional Riemann-Liouville ∂^α and the Caputo derivative D^α, which are defined in Definition 2.18 without specifying the domain. It is shown that the definition of ∂^α (with suitable domain) coincides with the fractional power of the derivation operator (Proposition 2.23). This is not a new result, just as the characterization of the domain of ∂^α in terms of Bessel potential spaces (Proposition 2.25). Literature should have been cited here, see my comments below. As to the Caputo derivative, the precise domain is not defined. I could only find the space ${}_0W^1_p(0, L)$ (see Remark 2.5), which is not a natural choice. Taking this space as domain leads, for example, to a problem in the first relation in Proposition 2.28 as $I^\alpha f$ does not have enough regularity in general. Some auxiliary results, such as e.g. Lemma 2.34, were taken from Ms

Ryszewska's own publications. The presentation of the derivation of the two fractional Stefan models essentially follows considerations by Voller (see [32], [33] in the dissertation). In the time-fractional model, the obtained boundary condition (2.35) (under more regularity) seems to be new.

Chapter 3 studies analytic properties of the operator $\partial_x D^\alpha$ defined on a domain $\mathcal{D}$ in $L^2(0, L)$ as well as the corresponding time-dependent diffusion equation. Ms Ryszewska first considers the case of mixed boundary conditions $u(1) = 0$ and $u_x(0, x) = 0$ (the latter only for $\alpha > 1/2$). The main result here is Theorem 3.5, which says that the operator $\partial_x D^\alpha$ generates an analytic C_0-semigroup. The proof uses classical results from semigroup theory (like the Lumer-Philips theorem) and spectral theory (numerical range) combined with special estimates for the operator $\partial_x D^\alpha$, in particular a suitable coercivity estimate (cf. Proposition 2.31). Having established generation of an analytic semigroup, further regularity and perturbation results are obtained by abstract theory. Next, Ms Ryszewska studies the case with Dirichlet boundary conditions and obtains a corresponding generation result (Theorem 3.10), where the domain of the generator has to be modified appropriately. It is further shown that the situation with non-homogeneous boundary conditions can be reduced to the already established result for the homogeneous case. This is no longer possible in the case of a non-homogeneous Neumann boundary condition, where the reduction step leads to a source term which in general is not square integrable. In order to solve the problem, Ms Ryszewska changes her framework and successfully studies existence and uniqueness in a weak setting via energy estimates and approximation. The corresponding results (Theorem 3.13 and Theorem 3.14) further enrich the theory for the diffusion equation under study inasmuch as they allow to treat less regular data.

Chapter 4 is devoted to the study of the space-fractional, one-dimensional Stefan problem. Here the difficulty consists in that another unknown comes in, the (time-dependent) position $s(t)$ of the interface separating the liquid from the solid phase. The dynamics of $s(t)$ depends on the fractional Caputo derivative $D^\alpha u$ of the temperature u at the interface. Ms Ryszewska uses the classical approach to such moving interface problems by transforming the model to a problem with fixed boundary. The price to pay here is that the transformed problem becomes quasilinear as in the equation for the temperature, coefficients appear which depend on s and its derivative, respectively. Considering s as a known sufficiently smooth quantity, the linear problem for the temperature is first solved by means of the theory of evolution operators and the contraction mapping principle in the framework of mild solutions (Theorem 4.2). Additional regularity is then obtained in Lemma 4.3 However, this regularity is still not enough to allow application of the maximum principle to the equation for s, which requires at least two derivatives in the Sobolev sense of the temperature u. Ms Ryszewska has to work hard to extract higher spatial regularity of u. The required estimates, which take many pages, are technically demanding. This section is the most challenging part of the dissertation. The results on the temperature problem are summarized in Theorem 4.9. Ms Ryszewska's approach to solve the full Stefan problem heavily relies on the maximum principle for the operators D^α and $\partial_x D^\alpha$, which is provided in the Lemmas 4.10, 4.11

and 4.12. These results are extended or improved versions of known results. Concerning Lemma 4.10, I wonder whether the assumption that f attains its local maximum can be dropped; it seems to be enough to assume that at x_0 one has a global maximum on $[0, x_0]$. Another important auxiliary result is the Hopf type lemma proved in Lemma 4.15. With all these preparations at hand, the main result of this chapter, but also of the whole dissertation, Theorem 4.1, is proved. Under certain conditions on the initial temperature, it establishes the global existence of a unique classical solution to the space-fractional Stefan problem. The proof uses Schauder's fixed point theorem to obtain existence; uniqueness is proved by means of monotonicity properties. The basic idea of the proof is inspired by Andreucci (see ref. [1] in the dissertation); however, as Chapters 3 and 4 show, its adaption to the space-fractional case is a very challenging problem. Ms Ryszewska has mastered this problem in an impressive manner. In the final part of Chapter 4, Ms Ryszewska constructs a self-similar solution for the space-fractional Stefan problem. The same result has been obtained independently in ref. [26]. Introducing a natural similarity variable, which reflects the scaling properties of the diffusion equation, the problem is reduced to solving a certain integral equation. Ms Ryszewska is able to derive a representation for the solution involving an infinite series.

Chapter 5 deals with the time-fractional version of the one-phase Stefan problem. The main result is Theorem 5.1, which establishes the existence of a self-similar solution and provides an integral representation for it. The approach is similar to that in Chapter 4, but due to the presence of the fractional time-derivative the problem is more involved. The chapter concludes with a nice result about the convergence of the obtained solution to a solution of the classical Stefan problem as the time order α goes to 1.

Summarizing, Ms Ryszewska has written a very good doctoral thesis on a current topic in modern Analysis. She uses a wide spectrum of methods like functional analytic tools, PDE methods and fractional calculus. A significant part of her results has already been published (see ref. [27] and [28]) or will appear (see [14]) in very good journals. Two of these papers are single-authored, [14] is a joint work with A. Kubica. This clearly shows Ms Ryszewska's ability to conduct independent research work. I am sure that her dissertation will become an important reference in the field of fractional diffusion/heat equations. Ms Ryszewska has worked very carefully, the work is very readable. There are virtually no typos.

Critically, I must note that too little literature is cited with regard to fractional derivatives and fractional diffusion equations. The list of references contains only 34 items, which is quite little for a doctoral thesis. Ms Ryszewska could have added more general background on fractional diffusion/heat equations including e.g. a comparison with the "classical" space-fractional diffusion equation with a fractional Laplacian. Fractional Riemann-Liouville and Caputo derivatives are more often found in time-fractional problems, which are closely related to Volterra equations. Concerning physics, the paper by Metzler and Klafter (2000) provides a very good motivation, as to the aspect of Volterra equations, the monograph of Prüss should be mentioned; it also includes considerations concerning the characterization of the domain of the fractional derivative, see also my paper in JEE (2005) and the Ph.D. thesis of Bazhlekova entitled "Fractional Evolution Equations in Banach spaces" (2001).

I strongly recommend that Ms Ryszewska's thesis be accepted as a doctoral dissertation.

Prof. Dr. Rico Zacher, Ulm, 24/03/2021